

# MODELLING COSMIC ACCELERATION IN MODIFIED YANG - MILLS THEORY

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## Abstract

We investigate the possibility that the modified Yang-Mills theories can produce an accelerated cosmic expansion. We take into account some specific non-trivial solution of the modified Yang-Mills equation obtained by the author earlier, which allows us to build several modifications of accelerated cosmic expansion.

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## 1 Introduction

Presently it is accepted by the scientific community that the Universe is experiencing an accelerated expansion. This is supported by many cosmological observations, such as SNe Ia [1], WMAP [2], SDSS [3] and X-ray [4]. These observations suggest that the Universe is dominated by the Dark Energy (DE), which provides the dynamical mechanism for the accelerated expansion of the Universe. Moreover, they suggest that the DE equation-of-state (EoS) parameter,  $\gamma = \frac{p}{\rho}$  might have crossed the phantom divide  $\gamma = -1$  from above in the near past. In order to explain this phenomena, one can either consider theories of modified gravity, or field models of DE. The simplest candidate of DE is a tiny positive time-independent cosmological constant, for which  $\gamma = -1$ . However, it is difficult to understand why the cosmological constant is about 120 orders of magnitude smaller than its natural expectation (the Planck energy density). This is the so-called cosmological constant problem. Another puzzle of DE is the cosmological coincidence problem: why are we living in an epoch in which the dark energy density and the dust matter energy are comparable? As a possible solution to these problems various dynamical models of DE have been proposed, such as quintessence [5]. So far, a large class of scalar-field DE models have been studied, including tachyon [6], ghost condensate [7] and quintom [8], and so forth. In addition, other proposals on DE include interacting DE models [9], braneworld models [10], and holographic DE models [11], etc. The quintom scenario of DE is designed to understand the nature of DE with gamma across -1. The quintom models of DE differ from the quintessence, phantom and k-essence and so on in the determination of the cosmological evolution.

Another class of DE models is based on the conjecture that a vector field can be the origin of DE [12],[13]. The YM field can be a kind of candidate for such a vector field [14],[15]. At the same time, it is well known that a pure YM field (with its EoS  $\gamma = 1/3$ ) can not provide accelerated expansion of the Universe, for which  $\gamma < 1/3$  is required. This is a direct consequence of conformal symmetry of the Lagrangian for a massless YM field. Any violations of conformal symmetry (e.g., as a result of quantum corrections [16] or of non-minimal coupling to gravity [17]) give a good chance for involving YM fields in reconstruction of DE. The alternative method for YM fields to be involved in DE problem is consideration of some sort of modified YM theory [18]-[20]. In this paper, we turn our attention to the issue of the YM fields as a source of DE in the frame of a modified YM theory. We aimed at deriving the necessary conditions for the possibility of those models to explain the accelerated expansion of the Universe. We also derive the corresponding equations for the scale factor evolution, and briefly discuss several examples for the DE models of this kind.

## 2 Equations of model dynamics

Let us consider the following action [20]:

$$S = - \int d^4x \sqrt{-g} \left\{ \frac{R - 2\Lambda}{2\kappa} + \Phi(F_{ik}^a F^{aik}) \right\}, \quad (1)$$

where  $F_{ik}^a = \partial_i W_k^a - \partial_k W_i^a + f^{abc} W_i^b W_k^c$ , and  $\Phi$  is a continuously differentiable function. Variation of (1) with respect to  $g^{ik}$  yields the Einstein equation

$$G_{ik} \equiv R_{ik} - \frac{1}{2} g_{ik} R = 2\kappa \left( \frac{1}{2} g_{ik} \Phi - 2\Phi' F_{ij}^a F_k^{aj} \right), \quad (2)$$

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where  $\Phi' \equiv d\Phi(I)/dI$ , and  $I = F_{ik}^a F^{aik}$  is the invariant of the Yang-Mills fields. As it follows from the action (1), the equation of motion for the field potential  $W_i^a$  turns into

$$\partial_i \left( \sqrt{-g} \Phi' F^{aik} \right) + \sqrt{-g} \Phi' \epsilon^{abc} W_i^b F^{cik} = 0. \quad (3)$$

We assume that the Universe is described by a Friedmann-Robertson-Walker (FRW) geometry:

$$ds^2 = N(t)^2 dt^2 - a^2(t)(dr^2 + \xi^2(r)d\Omega^2), \quad (4)$$

where  $\xi(r) = \sin r, r, \sinh r$  for the sign of space curvature  $k = +1, 0, -1$ , consequently. To study a FRW solution of the equations (2),(3), we can directly insert this metrics into action (1). Whereupon we obtain the following effective Lagrangian density (per unit solid angle):

$$L_{eff} = \frac{3}{8\pi G} \left( -\frac{a\dot{a}^2}{N} + kaN - \frac{\Lambda a^3}{3} N \right) \xi^2 - \Phi(I) a^3 N \xi^2. \quad (5)$$

At the same time, the generalized Wu-Yang ansatz for the  $SO_3$  YM fields can be written as [21]

$$W_0^a = x^a \frac{W(r, t)}{er},$$

$$W_\mu^a = \varepsilon_{\mu ab} x^b \frac{K(r, t) - 1}{er^2} + \left( \delta_\mu^a - \frac{x^a x_\mu}{r^2} \right) \frac{S(r, t)}{er}.$$

We can make the following substitution into this ansatz [22]:

$$K(r, t) = P(r) \cos \alpha(t), \quad S(r, t) = P(r) \sin \alpha(t), \quad W(r, t) = \dot{\alpha}(t).$$

As a result, we have the following formulae for the YM strength tensor components:

$$\mathbf{F}_{01} = \mathbf{F}_{02} = \mathbf{F}_{03} = 0, \quad \mathbf{F}_{12} = e^{-1} P'(r) (\mathbf{m} \cos \alpha + \mathbf{l} \sin \alpha),$$

$$\mathbf{F}_{13} = e^{-1} P'(r) \sin \theta (\mathbf{m} \sin \alpha - \mathbf{l} \cos \alpha), \quad \mathbf{F}_{23} = e^{-1} \sin \theta (P^2(r) - 1) \mathbf{n}, \quad (6)$$

where  $\mathbf{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ ,  $\mathbf{l} = (\cos \theta \cos \phi, \cos \theta \sin \phi, \cos \theta)$  and  $\mathbf{m} = (-\sin \phi, \cos \phi, 0)$  are the orthonormalized isoform vectors, and the prime means a derivative with respect to  $r$ . As noted in [22], the YM field (6) has only magnetic components. It is easy to find from (4) and (6) that the YM field invariant  $I = F_{ik}^a F^{aik}$  becomes as follows:

$$I = \frac{2}{e^2 a^4 \xi^2} \left[ 2P'^2 + \frac{(P^2 - 1)^2}{\xi^2} \right]. \quad (7)$$

Varying the effective Lagrangian density (5) over  $P(r)$ , and taking into account (7) we obtain the following Euler-Lagrange equation instead of YM equation (3):

$$\left\{ P'' - \frac{(P^2 - 1)P}{\xi^2} \right\} \Phi' + P' \Phi'' \frac{2}{e^2 a^4} \left\{ \frac{1}{\xi^2} \left[ 2P'^2 + \frac{(P^2 - 1)^2}{\xi^2} \right] \right\}' = 0, \quad (8)$$

where  $\Phi' \equiv d\Phi(I)/dI$ ,  $\Phi'' \equiv d^2\Phi(I)/dI^2$ .

The nontrivial solution for the YM equation obtained in [22],  $P(r) = \xi'(r) = \cos r, \cosh r$  for  $k = +1, -1$ , consequently, satisfies equation (8). Indeed, this solution turns both additive terms in the left-hand-side of (8) to zero. As it follows from (7), the valuable feature of this solution is that the YM invariant built on it depends only on time:

$$I = I(t) = \frac{6}{e^2 a^4(t)}. \quad (9)$$

The Hamiltonian constraint for (5) is

$$\left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3} \Phi(I) + \frac{\Lambda}{3}, \quad (10)$$

where  $I$  should be replaced with its value (9). By variation of (5) over  $a(t)$  with the subsequent choice of the gauge  $N = 1$ , one can obtain the following Friedmann equation:

$$2\frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = 8\pi G \Phi(I) - \frac{64\pi G}{e^2 a^4} \Phi'(I) + \Lambda. \quad (11)$$

### 3 Accelerated expansion

First we have to note that instead of equation (11) one frequently uses the following equation:

$$\frac{\ddot{a}}{a} = \frac{8\pi G}{3} \left[ \Phi(I) - 2I\Phi'(I) \right] + \frac{\Lambda}{3}, \quad (12)$$

which can be obtained by combining two equations (10), (11), and with the help of (9). This equation is just the differential consequence of equation (10). Nevertheless, it is convenient for investigation of the accelerated expansion.

Comparing equations (10) and (12) with the similar ones of the standard FRW cosmology of a perfect fluid, we can find the following expressions for the effective energy density and pressure of the YM field and the cosmological constant:

$$\rho = \Phi(I) + \frac{\Lambda}{8\pi G}, \quad p = -\Phi(I) + \frac{4}{3} I \Phi'(I) - \frac{\Lambda}{8\pi G}, \quad (13)$$

where the last terms are just the energy density and pressure associated with cosmological constant  $\Lambda$  with EoS:  $\rho_\Lambda = -p_\Lambda = \frac{\Lambda}{8\pi G}$ . Therefor, the EoS for the YM field and cosmological constant in our model is

$$\gamma = -1 + \frac{4}{3} \frac{I \Phi'(I)}{\Phi(I) + \frac{\Lambda}{8\pi G}}. \quad (14)$$

As it follows from equation (12) or, equivalently, from the inequality  $\gamma < -1/3$  in (14), the accelerating regime is possible if

$$\Phi(I) - 2I\Phi'(I) + \frac{\Lambda}{8\pi G} > 0. \quad (15)$$

Now we are going to consider some interesting examples concerning the standard and modified YM theories.

**a)** For the standard YM theory  $\Phi(I) = \frac{1}{16\pi}I$ , that is  $\Phi(I) = \frac{3}{8\pi e^2 a^4(t)}$ . Plugging this  $\Phi(I)$  into (14) we have

$$\gamma = -1 + \frac{4}{3} \left( 1 + \frac{\Lambda e^2}{3G} a^4(t) \right)^{-1}. \quad (16)$$

This gives the EoS  $\gamma = 1/3$  in the case of vanishing  $\Lambda$  as it must be for the pure radiation. Besides, the accelerating condition (15) turns into

$$a(t) > a_c = (3G/\Lambda e^2)^{1/4} \quad (17)$$

when  $\Lambda \neq 0$ . In this case, EoS (16) goes from  $1/3$  to  $-1$  during the evolution of the scale factor  $a(t)$ , and becomes less than  $-1/3$  as (17) is satisfied. Plugging  $\Phi(I) = \frac{3}{8\pi e^2 a^4(t)}$  into (10) we have the following equation for the scale factor:

$$\dot{a}^2 + k = \frac{G}{e^2 a^2} + \frac{\Lambda}{3} a^2,$$

which can be easily integrated. It should be noted that the similar equation was discussed earlier in [17].

**b)** Let us now suppose the phenomenological power-law dependence of  $\Phi(I)$  on  $I$ :  $\Phi(I) = A I^n$ , where  $A, n$  are some nonzero constants. In this case, one can rewrite inequality (15) as

$$(1 - 2n) A I^n + \frac{\Lambda}{8\pi G} > 0. \quad (18)$$

The latter means  $n < 1/2$  in the case of vanishing cosmological constant:  $\Lambda = 0$ . At the same time, according to (14) the EoS becomes  $\gamma = -1 + \frac{4}{3}n = \text{constant}$ . For  $n < 1/2$ ,  $\gamma < -\frac{1}{3}$  that is this model experiences eternal accelerated expansion.

Let us revert to the case of non-zero  $\Lambda$ . As it follows from (9) and (14),

$$\gamma_n = -1 + \frac{4}{3} n \left( 1 + \frac{\Lambda e^{2n}}{8\pi G A 6^n} a^{4n}(t) \right)^{-1}. \quad (19)$$

With the help of (18), it is easy to show that  $\gamma_n < -1/3$  in this case too. Of course, the particular case  $n = 1$ ,  $A = 1/16\pi$  leads to (16). Now plugging  $\Phi(I) = A \frac{6^n}{e^{2n} a^{4n}(t)}$  into (10) we have the following equation for the scale factor:

$$\dot{a}^2 + k = B a^{2(1-2n)} + \frac{\Lambda}{3} a^2, \quad (20)$$

where  $B = 8\pi G A 6^n / 3e^{2n}$ . This equation can be integrated for several  $n$  in an explicit form.

**c)** Now we consider the case of a widely discussed non-Abelian Born-Infeld (BI) Lagrangian (see, e.g., [17] and bibliography therein):

$$L_{NBI} = \frac{\beta^2}{4\pi} \left( \sqrt{1 + \frac{F_{ik}^a F^{aik}}{\beta^2} - \frac{(\tilde{F}_{ik}^a F^{aik})^2}{16\beta^4}} - 1 \right),$$

where  $\beta$  is the critical BI field strength,  $\tilde{F}_{ik}^a$  is a dual YM strength tensor. From (6), we can find that for our solution the second invariant of YM field  $\tilde{\mathbf{F}}_{ik} \mathbf{F}^{ik} = 0$ . Hence, we can identify  $\Phi(I)$  with

$$\Phi(I) = \frac{1}{16\pi\alpha} \left( \sqrt{1 + 2\alpha I} - 1 \right),$$

where  $\alpha = 1/2\beta^2$ . Under our solution for YM field, this model goes from  $\gamma = -1/3$  to  $\gamma = -1$  according to the following equation:

$$\gamma_{BI} = -1 + \frac{8\alpha}{e^2} \left( a^4 + \frac{12\alpha}{e^2} \right)^{-1/2} \left[ \sqrt{a^4 + \frac{12\alpha}{e^2}} + \left( \alpha \frac{\Lambda}{G} - 1 \right) a^2 \right]^{-1}. \quad (21)$$

Simultaneously, the scale factor is driven by the equation

$$\dot{a}^2 + k = \frac{G}{3\alpha} \left( \sqrt{a^4 + \frac{12\alpha}{e^2}} - a^2 \right) + \frac{\Lambda}{3} a^2. \quad (22)$$

It should be noted that the same equation was obtained earlier in non-linear BI theory on the brane [17].

**d)** At last, let us consider the effective YM field cosmic model based on the effective Lagrangian up to 1-loop order [16], [17], [23]. In our notation, this Lagrangian can be written as follows:

$$\Phi(I) = -\frac{b}{4} I \ln \left| \frac{I}{2\kappa^2} \right|, \quad (23)$$

where  $\kappa$  is the renormalization scale of dimension of squared mass,  $b = 11/12\pi^2$  is the Callan-Symanzik coefficient for the generic gauge group considered here. From (21), we can find that  $\Phi' = -(b/4) (\ln |I/2\kappa^2| + 1)$ . Due to this formula together with (9) and (14), we obtain the following EoS:

$$\gamma_{YMC} = \frac{1}{3} - \frac{4}{3} \frac{1 + \frac{\Lambda e^2}{12\pi G b} a^4(t)}{\ln \left| \frac{(e\kappa)^2}{3} a^4(t) \right| + \frac{\Lambda e^2}{12\pi G b} a^4(t)}, \quad (24)$$

which displays more complicated behavior on time then the ones considered above. Indeed, it starts from  $\gamma_{YMC} = 1/3$  at  $a = 0$  but approaches  $\gamma_{YMC} = -1$  through the break of its continuity at  $a(t) = a_{cr}$ , which can be found from the following transcendent equation:

$$a_{cr}^4(t) = \frac{3}{(e\kappa)^2} \exp \left\{ -\frac{\Lambda e^2}{12\pi G b} a_{cr}^4(t) \right\}.$$

Nevertheless, if the model starts its expansion at  $a_0 > a_{cr}$  (non-singular model), then no problem of that kind occurs. Taking into account (9), (10) and (23) we can find that the scale factor of the model is driven by the following equation:

$$(\dot{a})^2 + k = \frac{4\pi G b}{e^2 a^2} \ln \left| \frac{(e\kappa)^2 a^4(t)}{3} \right| + \frac{\Lambda}{3} a^2. \quad (25)$$

In the case of vanishing  $\Lambda$ , the EoS of this model follows from (24) as

$$\gamma_{YMC}^0 = \frac{1}{3} - \frac{4}{3} \left( \ln \left| \frac{(e\kappa)^2 a^4(t)}{3} \right| \right)^{-1}.$$

The critical value  $a(t)$  becomes  $a_{cr}^0 = (3/e^2\kappa^2)^{1/4} \neq 0$ .

Finally, we have to note that EoS (19), (21), (24) and corresponding equations (20), (22), (25) derived in the sub-sections **b)**, **c)** and **d)** consequently can be investigated in more details. That could be done analytically or, in any case, numerically. We do not make it our aim in this short communication.

## 4 Conclusion

In summary, the standard and some modified YM theories in FRW cosmology are studied in this paper. The specific non-trivial solution of the modified YM equation (8) proposed by the author earlier allows us to build several modifications of accelerated cosmic expansion. All of them possess EoS  $\gamma \sim -1$  at late time, so the cosmic coincidence problem can be avoided in those models. Besides, we have derived the equations for the cosmic scale factor in all those models. In our opinion, more significant result of our study is that we can now to consider the wide range of modified YM theories in cosmology. For such a purpose, the equations (9,10), (13,14) and inequality (15) have to be employed. Further details and consequences of the modified YM models considered here are in progress.

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